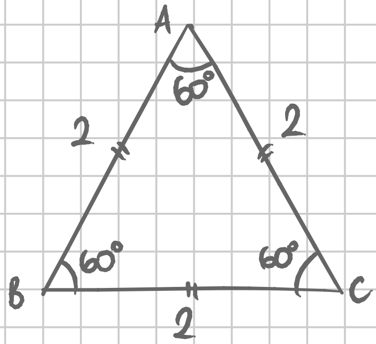
C2 Exercise 8D (exact values of trig functions)

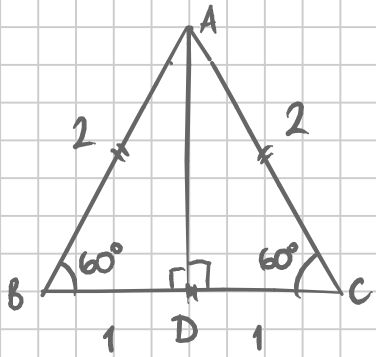
Note Title 28/04/2013

Before we start let's go through the two standard triangles.

① Start with an equilateral triangle. Give it side length 2 units (this will make the numbers easier later)

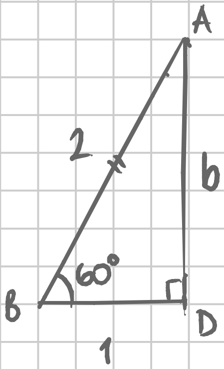


Now 'drop a perpendicular from A to meet BC at D'



... this gives you two right-angled triangles $\triangle ABD$ and $\triangle ACD$. (do you see why we made the side length 2 now?)

Let's focus on $\triangle ABD$:



... we can use Pythagoras theorem on this to work out the height AD which I'll call 'b'.

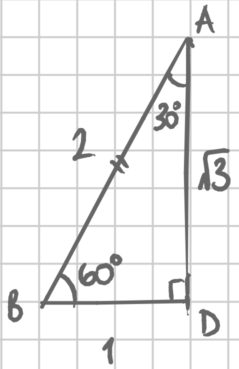
$$b^2 + 1^2 = 2^2$$

$$\Rightarrow \quad b^2 + 1 = 4$$

$$\Rightarrow \quad b^2 = 3$$

$$\Rightarrow \quad b = \sqrt{3}$$

And it's also obvious that angle $B\hat{A}D = 30°$...



Applying the standard trigonometry to this:

$$\sin 60^\circ = \frac{opp}{hyp} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{adj}{hyp} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{opp}{adj} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

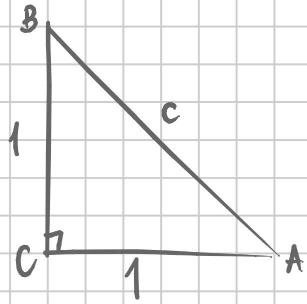
And similarly looking at the triangle from the perspective of the 30°:

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

and $\tan 30 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$

② Start with an isosceles-right-triangle with two sides of length 1 unit.



We can apply Pythagoras' theorem again to find length AB which I have labelled c

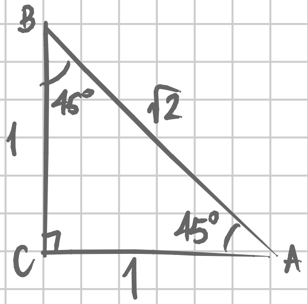
$$1^2 + 1^2 = c^2$$

$$\Rightarrow \quad c^2 = 2$$

$$\Rightarrow \quad c = \sqrt{2}$$

It should be obvious that $C\hat{B}A = C\hat{A}B = 45°$.

So we have the triangle:



Applying the standard trig ratios

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and $$\tan 45 = \frac{1}{1} = 1.$$

You are expected to:

(i) memorise sin, cos and tan of 30°, 45° and 60° in these exact (surd) forms

(ii) be able to prove them using a version of the arguments above

(iii) be able to apply them to derive exact values for $\sin 135^\circ$, $\cos(-60)^\circ$, $\tan 390^\circ$, etc.

Learn:

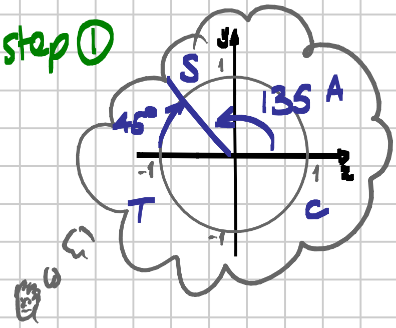
$\theta =$	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

1 Express the following as trigonometric ratios of 30°, 45° or 60° and hence find their exact values.

a $\sin 135^\circ = \sin 45^\circ$

$= \frac{\sqrt{2}}{2}.$

step ①

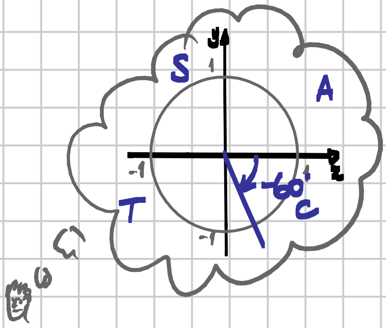


step ② you need to recall the standard results from above:

$\sin 45 = \frac{\sqrt{2}}{2}.$

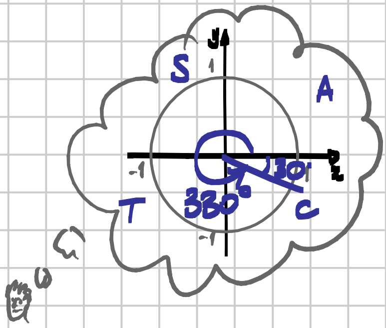
b $\sin -60^\circ = -\sin 60^\circ$

$= -\frac{\sqrt{3}}{2}$



c $\sin 330^\circ = -\sin 30^\circ$

$= -\frac{1}{2}$



d $\sin 420^\circ = \sin(420 - 360)$

$= \sin 60^\circ$

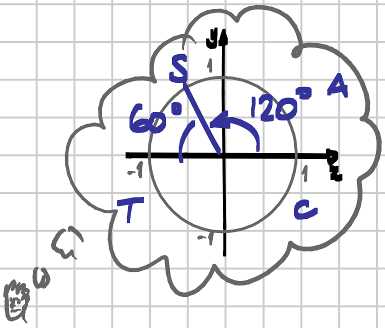
$= \frac{\sqrt{3}}{2}.$

e $\sin(-300)^\circ = \sin 60^\circ$

$= \frac{\sqrt{3}}{2}$

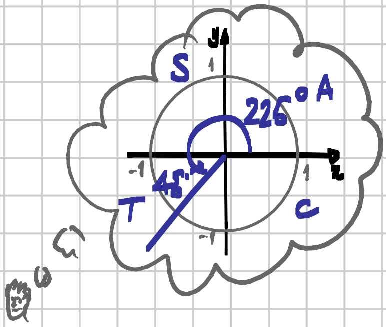
f $\cos 120^\circ = -\cos 60^\circ$

$= -\frac{1}{2}$



1 g $\cos 300^\circ = \cos 60^\circ$
$= \frac{1}{2}$

h $\cos 225^\circ = -\cos 45^\circ$
$= -\frac{\sqrt{2}}{2}.$



i $\cos(-210)^\circ = -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}.$

j $\cos 495^\circ = \cos(495 - 360)$
$= \cos 135^\circ$
$= -\cos 45^\circ$
$= -\sqrt{2}/2$

k $\tan 135^\circ = -\tan 45^\circ$
$= -1$

l $\tan(-225)^\circ = \tan 135^\circ$
$= -1$ (see above)

m $\tan 210^\circ = \tan 30^\circ$
$= \frac{\sqrt{3}}{3}.$

n $\tan 300^\circ = -\tan 60^\circ$
$= -\sqrt{3}$

o $\tan(-120)^\circ = \tan 60^\circ$
$= \sqrt{3}$

2 In section 8.3 we found that

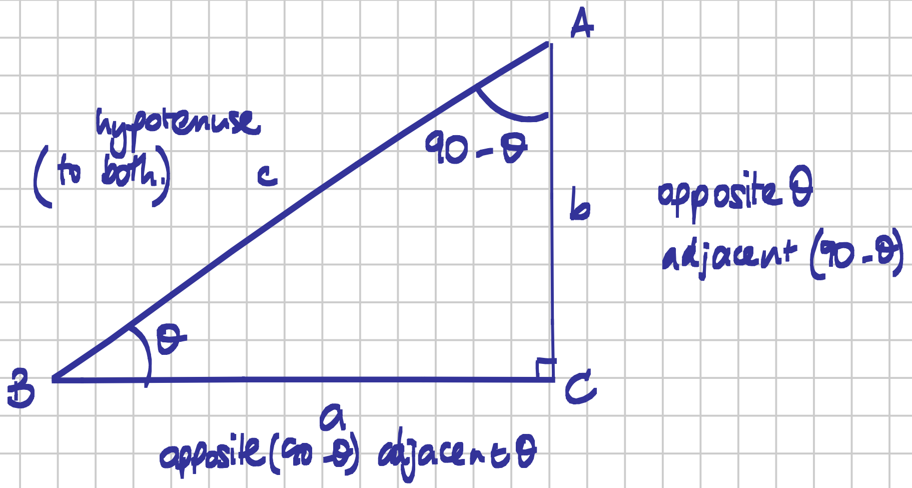
$$\sin 30° = \cos 60°$$
$$\cos 30° = \sin 60°$$
and $$\tan 60° = \frac{1}{\tan 30°}$$

These are particular examples of a more general result that:

$$\sin(90-\theta) = \cos\theta$$
$$\cos(90-\theta) = \sin\theta$$
and $$\tan(90-\theta) = \frac{1}{\tan\theta}$$

where θ is an angle measured in degrees.

Use a right angled triangle ABC to verify these results for the case where θ is acute.



$\sin\theta = \frac{opp}{hyp} = \frac{b}{c}$ but $\cos(90-\theta) = \frac{adj}{hyp} = \frac{b}{c}$

so $\sin\theta = \cos(90-\theta)$

similarly $\cos\theta = \sin(90-\theta)$

and $\tan(90-\theta) = \frac{a}{b}$ where $\tan\theta = \frac{b}{a}$

so $\tan(90-\theta) = \frac{1}{\tan\theta}$